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Obliquely interacting solitary waves

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Nonlinear oblique interactions between two slightly dispersive gravity waves (in particular, solitary waves) of dimensionless amplitudes α_1 and α_2 (relative to depth) and relative inclination 2ψ (between wave normals) are classified as *weak* if $\sin^2\psi \gg \alpha_{1,2}$ or *strong* if $\psi^2 = O(\alpha_{1,2})$. Weak interactions permit superposition of the individual solutions of the Korteweg–de Vries equation in first approximation; the interaction term, which is $O(\alpha_1\alpha_2)$, then is determined from these basic solutions.

Strong interactions are intrinsically nonlinear. It is shown that these interactions are phase-conserving (the sum of the phases of the incoming waves is equal to the sum of the phases of the outgoing waves) if $|\alpha_2-\alpha_1| > (2\psi)^2$ but not if $|\alpha_2-\alpha_1| < (2\psi)^2$ (e.g. the reflexion problem, for which the interacting waves are images and $\alpha_2 = \alpha_1$). It also is shown that the interactions are singular, in the sense that regular incoming waves with sech^2 profiles yield singular outgoing waves with $-\text{csch}^2$ profiles, if

$$\psi_- < |\psi| < \psi_+, \quad \text{where} \quad \psi_\pm = \tfrac{1}{2}|(3\alpha_2)^{\frac{1}{2}} \pm (3\alpha_1)^{\frac{1}{2}}|.$$

Regular interactions appear to be impossible within this singular regime, and its end points, $|\psi| = \psi_\pm$, are associated with resonant interactions.

1. Introduction

A striking property of solitons (soliton $\equiv$ Boussinesq's solitary wave in the present context) is their asymptotic preservation of form following an interaction. Two distinct types of one-dimensional interaction have been studied: (i) two or more solitons of different strengths move in the same direction, interact for a relatively long time, and emerge with phase shifts that are $O(1)$ as $\alpha \downarrow 0$, where

$$\alpha = \text{free-surface amplitude/quiescent depth}; \tag{1.1}$$

(ii) two solitons move in opposite directions, interact for a relatively short time, and emerge with phase shifts that are $O(\alpha)$. The first type of interaction, which has been widely studied [see Scott, Chu & McLaughlin (1973) and Whitham (1974)† for references], is governed by the Korteweg–de Vries (KdV) equation and may be analytically characterized as essentially nonlinear, or *strong*. The second class, which includes reflexion at a rigid wall, may be analytically

† Whitham's treatise is referred to throughout the subsequent exposition by the prefix W, followed by the appropriate section or equation number.

characterized as *weak* in the sense that the first approximation to the solution may be obtained simply by superimposing the solutions of the KdV equation for the individual solitons. It appears to have been studied originally by Gwyther (1900) and subsequently by Meyer (1963), Benney & Luke (1964), Byatt-Smith (1971), Oikawa & Yajima (1973), Chen (1975) and Satsuma (1976).

Gwyther (1900) obtained, but did not integrate, a result that is equivalent to (3.7) below for the special case of reflexion and inferred from it that the interaction term is of second order.

Meyer's (1963) discussion is based on the Boussinesq equation, which is inapplicable to waves travelling in different directions (see remarks by Lin & Clark 1959; Long 1964; Byatt-Smith 1971). Nevertheless, Meyer's qualitative conclusion, that the solutions of the KdV equation for oppositely directed solitary waves may be superimposed in first approximation, is correct. Oikawa & Yajima (1973) also obtain results for solitary-wave reflexion from the Boussinesq equation, although they appear to recognize the inapplicability of these results to water waves.

Benney & Luke (1964) give a systematic formulation of the perturbation problem for two-dimensional weak interactions of the type considered here; however, as pointed out by Byatt-Smith, their end result for the reflexion problem is incorrect.

Byatt-Smith (1971) derives an integral-equation formulation of the boundary-value problem for one-dimensional surface waves without the prior invocation of scaling approximations. He then obtains a second approximation, including all $O(\alpha^2)$ terms, to its solution for the reflexion of a solitary wave.

Chen (1975) shows that the Bäcklund transformation previously applied to the one-dimensional problem (cf. W § 17.12) may be extended to interactions between two obliquely directed solitons, his q_1 and q_2, to obtain a 'wedged soliton', his q_3. He does not give an explicit representation of q_3; however, after making appropriate choices of the four parameters (two for each of q_1 and q_2) that are implicit in his result, I find that it is equivalent to (6.5) below.

Satsuma (1976) solves a 'two-dimensional KdV equation' to obtain equivalents of (6.5)–(6.7) below and gives a formal solution for an oblique interaction among N solitons. His asymptotic interpretation of his solution differs from that given in § 6 (see below).

I consider here the interaction between two obliquely moving solitons, starting (in § 2) from perturbation equations that are equivalent to those developed by Lin & Clark (1959), Benney & Luke (1964) and Whitham (W § 13.11). The general problem is characterized by three parameters, which comprise the amplitudes of the two basic waves and the relative inclination of their normals. Let c_n be the wave speed of the nth wave ($n = 1, 2$) and 2ψ the angle between the wave normals $\mathbf{n}_1$ and $\mathbf{n}_2$; then α [see (1.1)],

$$\epsilon = (c_2 - c_1)/(c_2 + c_1), \quad \kappa = \sin^2\psi \tag{1.2a, b}$$

are suitable measures of mean strength, relative strength and obliquity. It follows from the perturbation equations that an interaction is weak if $\kappa \gg \alpha$ or strong if $\kappa = O(\alpha)$ as $\alpha \downarrow 0$ (the adjective *strong* is used here in the sense of scattering theory, but it should be emphasized that the actual nonlinearity remains weak in the sense that the perturbation equations are valid). It also follows from the perturbation equations that

$$\epsilon = \tfrac{1}{4}(\alpha_2 - \alpha_1) + O(\alpha^2), \tag{1.3}$$

where $\alpha_{1,2}d$ (d = quiescent depth) are the maximum amplitudes of the incoming waves.

I give a general solution for $\kappa \gg \alpha$ in §3 that is similar in principle to that given by Benney & Luke (1964) and describes a weak interaction between any two solutions of the KdV equation. It includes reflexion at a rigid wall as a special case, with results that provide a simple generalization of those given by Byatt-Smith (1971) for normal reflexion of a solitary wave.

I then go on in §5 (after recapitulating the known results for a solitary wave in §4) to consider the problem of grazing reflexion, for which $\epsilon = 0$, $\psi_2 = -\psi_1 = \psi$ and $\kappa = O(\alpha)$, and obtain an explicit solution of the perturbation equations through an extension of a Riccati-type transformation applied previously by Hirota and Whitham (see §W17.2) to the one-dimensional problem. The principal result is that if the incoming solitary wave has the dimensionless free-surface displacement

$$\eta_1 = \operatorname{sech}^2\theta_1, \quad \theta_1 = k_1(\mathbf{n}_1\cdot\mathbf{x} - c_1 t), \tag{1.4a, b}$$

the reflected wave is given (asymptotically) by

$$\eta_{2\delta} = \operatorname{sech}^2(\theta_2 - \delta), \tag{1.5}$$

where

$$\delta = \tfrac{1}{2}\log\{\kappa/(\kappa - 3\alpha)\}, \tag{1.6}$$

and θ_2 is given by (1.4b) with $\mathbf{n}_1$ replaced by $\mathbf{n}_2$ (but $k_1 = k_2$ and $c_2 = c_1$). The reflected wave is singular according to $\eta \sim -\operatorname{csch}^2(\theta - \mathscr{R}\delta)$ if $\kappa < 3\alpha$, and regular reflexion then appears to be impossible (see penultimate paragraph in this section).

Reflexion may be regarded as a symmetrical interaction between the incoming wave η_1 and its image η_2 that yields the outgoing waves $\eta_{1\delta}$ and $\eta_{2\delta}$; in brief,

$$\{\eta_1, \eta_2\} \rightarrow \{\eta_{1\delta}, \eta_{2\delta}\}, \tag{1.7}$$

where $\eta_{n\delta}$ differs from η_n only by the negative phase shift δ. It is worth remarking that the interaction is stationary in a reference frame that moves parallel to the reflecting boundary with the speed $c_* = c_1 \sec\psi$ (see below).

I extend the solution of §5 to two interacting solitons of unequal strengths ($\epsilon \neq 0$) in §6 [Satsuma (1976) and, to a lesser extent, Chen (1975) have anticipated me in the basic solution (6.5) but not in the interpretation thereof (see below)]. The resolution among incoming and outgoing waves is less direct than for the reflexion problem in consequence of a spatial non-uniformity of the asymptotic limits $t \rightarrow \pm\infty$ with θ_n fixed; however, this difficulty is avoided by adopting a reference frame R_* in which the interaction is stationary. The speed and orientation of R_* are determined by the requirement that the projection of its velocity $\mathbf{c}_*$ on the wave normal of each of the interacting waves be equal to the phase velocity of that wave (cf. Snell's law):

$$\mathbf{n}_{1,2}\cdot\mathbf{c}_* = c_{1,2}. \tag{1.8}$$

Proceeding in this way, I find that the interactions are of the form

$$\{\eta_{1\delta}, \eta_2\} \to \{\eta_1, \eta_{2\delta}\}, \tag{1.9}$$

and therefore phase-conserving (the sum of the phases of the incoming waves is equal to the sum of the phases of the outgoing waves), if $\epsilon < -\tan^2\psi$ and similarly, with $1 \leftrightarrow 2$, if $\epsilon > \tan^2\psi$; they are of the form (1.7), and therefore not phase-conserving, if $|\epsilon| < \tan^2\psi$. Satsuma (1976) overlooks the spatial non-uniformity of the asymptotic limits $t \to \pm\infty$ with θ_n fixed and concludes that all interactions are phase-conserving, like those of (1.9) but unlike those of (1.7).

The phase shift for $\alpha_2 \neq \alpha_1$ is given by [cf. (1.6) for $\alpha_1 = \alpha_2$]

$$\delta = \tfrac{1}{2}\log\{(\kappa-\kappa_-)/(\kappa-\kappa_+)\}, \tag{1.10}$$

where

$$\kappa_\mp = \tfrac{3}{4}(\alpha_2^{\frac{1}{2}} \mp \alpha_1^{\frac{1}{2}})^2. \tag{1.11}$$

The interaction is regular if either $\kappa < \kappa_-$ or $\kappa > \kappa_+$, but it is singular if $\kappa_- < \kappa < \kappa_+$. (An arbitrary constant, not necessarily real, may be added to the phase θ_n. If this constant is determined such that either η_n or $\eta_{n\delta}$ is regular, then either $\eta_{n\delta}$ or η_n is singular if $\kappa_- < \kappa < \kappa_+$.) Both Chen (1975) and Satsuma (1976) appear to overlook the existence of the singular regime $\kappa_- < \kappa < \kappa_+$, although it is implicit in Satsuma's solution, as also are other such regimes for $N > 2$.

The full significance of the singular regime is not clear at this time. It is conceivable that the singular solutions could represent waves of depression in those parts of their domains in which they remain finite; however, it is known, from Scott Russell's (1844) experiments and from the consideration that nonlinearity and dispersion both reduce the speed of a depression of the free surface (whereas they have opposite effects, and are in balance, for a solitary wave), that a solitary wave of depression is impossible, and it seems more likely that the singular solutions are unstable and that a regular interaction between two solitary waves is impossible in the parametric domain $\kappa_- < \kappa < \kappa_+$. This conclusion is supported by field and laboratory observations (Wiegel 1964) that regular reflexion gives way to 'Mach reflexion' (geometrically similar to the corresponding shock-wave reflexion) for sufficiently small ψ, although the available data appear to be inadequate for quantitative comparison with the present results.†

It appears to be significant that the conditions $\kappa = \kappa_\pm$ are precisely those necessary for a resonant interaction among three solitary waves with $\mathbf{k}_3 = \mathbf{k}_2 \pm \mathbf{k}_1$ and $k_3c_3 = k_2c_2 \pm k_1c_1$ (the alternative signs are vertically ordered). I plan to discuss these interactions and their relevance to Mach reflexion in a subsequent paper.

2. Perturbation equations

Following Whitham (W § 13.11), we henceforth use only dimensionless variables. Let $l\mathbf{x} \equiv l\{x, z\}$ and dy be horizontal and vertical co-ordinates, $lt/(gd)^{\frac{1}{2}}$ the

† Moses (1976) obtains a solution of a two-dimensional KdV equation with the "possible interpretation . . . that it corresponds to the reflection of a wave by a wall, in which the incident wave is singular and the reflected wave is nonsingular but highly dispersive". There appears to be little or no relation between his solution and that for a conventional solitary wave.

time, $a\eta$ the free-surface displacement and $la(g/d)^{\frac{1}{2}}\phi$ the velocity potential, where d is the quiescent depth, $a \equiv \alpha d$ is a characteristic amplitude and $l \equiv d/\sqrt{\beta}$ is a characteristic wavelength; then the boundary-value problem for inviscid irrotational motion is described by

$$\beta\Delta\phi + \phi_{yy} = 0 \quad (0 < y < 1+\alpha\eta), \tag{2.1}$$

$$\phi_y = 0 \quad (y = 0), \tag{2.2}$$

$$\eta_t + \alpha\nabla\phi\cdot\nabla\eta - \beta^{-1}\phi_y = 0 \tag{2.3a}$$

and
$$\eta + \phi_t + \tfrac{1}{2}\alpha(\nabla\phi)^2 + \tfrac{1}{2}\alpha\beta^{-1}\phi_y^2 = 0 \quad (y = 1+\alpha\eta), \tag{2.3b}$$

where ∇ and $\Delta \equiv \nabla^2$ are the gradient and Laplacian operators in a horizontal plane, and subscripts imply partial differentiation.

Posing the solution of (2.1) and (2.2) in the form

$$\phi(\mathbf{x}, y, t) = \sum_0^\infty (-\beta\Delta)^m f(\mathbf{x}, t)\, y^{2m}/(2m)! \tag{2.4}$$

and eliminating η between (2.3 a, b) yields

$$\eta = -f_t - \tfrac{1}{2}\alpha(\nabla f)^2 + \tfrac{1}{2}\beta f_{ttt} + O(\alpha^2) \tag{2.5}$$

and
$$f_{tt} - \Delta f = -\alpha\{\tfrac{1}{2}f_t^2 + (\nabla f)^2\}_t + \tfrac{1}{3}\beta f_{tttt} + O(\alpha^2). \tag{2.6}$$

3. Weak interactions

A wave that is slowly varying in a reference frame moving with the basic wave speed at an angle ψ with respect to the x axis may be described by $f = F(\xi, \tau)$, where

$$\xi = \mathbf{n}\cdot\mathbf{x} - t, \quad \mathbf{n} = \{\cos\psi, \sin\psi\}, \quad \tau = \alpha t, \tag{3.1 a, b, c}$$

and F satisfies the KdV-like equation

$$2\alpha F_\tau + \tfrac{3}{2}\alpha F_\xi^2 + \tfrac{1}{3}\beta F_{\xi\xi\xi} + O(\alpha^2) = 0 \tag{3.2}$$

in first approximation. We proceed on the hypothesis that the interaction between two such waves may be described by

$$f(\xi_1, \xi_2, \tau) = F_1(\xi_1, \tau) + F_2(\xi_2, \tau) + \alpha F_{12}(\xi_1, \xi_2, \tau), \tag{3.3}$$

where ξ_n ($n = 1, 2$) is given by (3.1) with $\psi = \psi_n$, F_n is prescribed and satisfies (3.2), and F_{12} is to be determined. Note that only the first approximation to F_n is required for the determination of the first approximation to F_{12} but that F_n may comprise $O(\alpha)$ components that are not determined by (3.2).

Transforming (2.5) and (2.6) yields

$$\eta = (\partial_1 + \partial_2 - \alpha\partial_\tau)f - \alpha\{\tfrac{1}{2}(\partial_1 f)^2 + \tfrac{1}{2}(\partial_2 f)^2 + (1-2\kappa)(\partial_1 f)(\partial_2 f)\} - \tfrac{1}{2}\beta(\partial_1+\partial_2)^3 f + O(\alpha^2) \tag{3.4}$$

and

$$(\partial_1+\partial_2)\,[2\alpha\,\partial_\tau f + \alpha\{\tfrac{3}{2}(\partial_1 f)^2 + \tfrac{3}{2}(\partial_2 f)^2 + (3-4\kappa)(\partial_1 f)(\partial_2 f)\} + \tfrac{1}{3}\beta(\partial_1+\partial_2)^3 f] - 4\kappa\,\partial_1\partial_2 f + O(\alpha^2) = 0, \tag{3.5}$$

where
$$\kappa = \tfrac{1}{2}(1 - \mathbf{n}_1\cdot\mathbf{n}_2) = \sin^2\tfrac{1}{2}(\psi_1 - \psi_2) \tag{3.6}$$

and ∂_n implies partial differentiation with respect to ξ_n. Substituting (3.3) into (3.5) and requiring each of F_1 and F_2 to satisfy (3.2) yields the interaction equation

$$(3-4\kappa)(\partial_1+\partial_2)\,\partial_1 F_1 \partial_2 F_2 - 4\kappa\,\partial_1\partial_2 F_{12} + O(\alpha) = 0. \quad (3.7)$$

Integrating (3.7) yields†

$$F_{12} = (\tfrac{3}{4}\kappa^{-1}-1)(\partial_1+\partial_2)\,F_1F_2 + O(\alpha) \quad (3.8)$$

plus an arbitrary function of the form $G_1(\xi_1,\tau)+G_2(\xi_2,\tau)$, which (by definition) is comprised by the $O(\alpha)$ component of F_1+F_2. After substituting (3.8) into (3.3) and invoking Taylor's theorem, we obtain

$$f = F_1(\xi_1+\chi_2,\tau)+F_2(\xi_2+\chi_1,\tau)+O(\alpha^2), \quad (3.9)$$

where
$$\chi_n = \alpha(\tfrac{3}{4}\kappa^{-1}-1)\,F_n(\xi_n,\tau). \quad (3.10)$$

Substituting (3.9) into (3.4) and eliminating $\partial_\tau F_n$ with the aid of (3.2), we obtain

$$\eta = N_1(\xi_1+\chi_2,\tau)+N_2(\xi_2+\chi_1,\tau)+\alpha I N_1 N_2 + O(\alpha^2), \quad (3.11)$$

where
$$N_n = (\partial_n - \tfrac{1}{3}\beta\partial_n^3)\,F_n + \tfrac{1}{4}\alpha(\partial_n F_n)^2 + O(\alpha^2) \quad (n = 1,2) \quad (3.12)$$

and
$$I = \tfrac{3}{2}\kappa^{-1} - 3 + 2\kappa. \quad (3.13)$$

The interaction parameter I decreases from $\frac{1}{2}$ for $\kappa = 1$ to a minimum of 0·464 at $\kappa = 0{\cdot}866$ and then increases monotonically with decreasing κ; however, the hypothesis that the interaction is $O(\alpha)$ holds only for $\kappa \gg \alpha$, and the approximations (3.8)–(3.11) fail for $\kappa = O(\alpha)$.

The solution for the reflexion of a wave incident from $\{-\infty,\infty\}$ at an angle $-\psi$ with respect to a rigid wall at $z = 0$ is obtained by setting $\psi_2 = -\psi_1 = \psi$, $\kappa = \sin^2\psi$, $F_1 = F_2 \equiv F$ and $N_1 = N_2 \equiv N$ in (3.9)–(3.13), such that $\partial F/\partial z = 0$ at $z = 0$. The run-up at the wall is given by

$$\eta_0 = 2N(\xi_0+\chi_0,\tau)+\alpha I N^2(\xi_0,\tau)+O(\alpha^2), \quad (3.14)$$

where
$$\xi_0 = x\cos\psi - t, \quad \chi_0 = \alpha(\tfrac{3}{4}\kappa^{-1}-1)\,F(\xi_0,\tau). \quad (3.15a, b)$$

The maximum run-up is (after restoring dimensions) $2a + I(a^2/d)$ if a is the amplitude of the incident wave (such that $N_{max} \equiv 1$).

4. Solitary wave

The solitary wave is, by hypothesis, a function of the single phase variable

$$\theta = \mathbf{k}\cdot\mathbf{x} - \omega t + \theta_0 \equiv k[\xi+\xi_0-\{(c-1)/\alpha\}\tau], \quad (4.1)$$

where $\mathbf{k} \equiv k\mathbf{n}$ is the wavenumber, $\omega \equiv kc$ is the angular frequency, c is the wave speed and $\theta_0 \equiv k\xi_0$ is a phase constant. The solution for a wave of maximum amplitude $\alpha_n d$ then is given by (Laitone 1960)

$$\alpha\eta = \alpha_n(1-\tfrac{3}{4}\alpha_n\tanh^2\theta)\,\mathrm{sech}^2\,\theta + O(\alpha^3), \quad (4.2)$$

$$\beta^{\frac{1}{2}}k_n = (\tfrac{3}{4}\alpha_n)^{\frac{1}{2}}(1-\tfrac{5}{8}\alpha_n)+O(\alpha^2), \quad c_n = 1+\tfrac{1}{2}\alpha_n - \tfrac{3}{20}\alpha_n^2 + O(\alpha^3). \quad (4.3a, b)$$

The characteristic length l is arbitrary, and we choose $\beta = \frac{3}{4}\alpha$ below.

† The result (3.8) is equivalent to Benney & Luke's (1964) equation (34). They do not give counterparts of the general results (3.11)–(3.15), but their result (39) for the reflexion of a solitary wave differs significantly from (4.4) below.

Substituting (4.2) into (3.10)–(3.12) and choosing $F_n = 0$ at $\theta_n = \infty$ yields (after re-normalizing χ_n)

$$\alpha\eta = \alpha_1(1-\tfrac{3}{4}\alpha_1\tanh^2\theta_1)\operatorname{sech}^2(\theta_1+\chi_2)+\alpha_2(1-\tfrac{3}{4}\alpha_2\tanh^2\theta_2)\operatorname{sech}^2(\theta_2+\chi_1) + \alpha_1\alpha_2 I\operatorname{sech}^2\theta_1\operatorname{sech}^2\theta_2+O(\alpha^2), \quad (4.4)$$

where

$$\chi_n = (\alpha_1\alpha_2)^{\frac{1}{2}}(1-\tfrac{3}{4}\kappa^{-1})(1-\tanh\theta_n)+O(\alpha^2). \quad (4.5)$$

5. Strong symmetric interactions

We now consider the interaction between two solitary waves with $\alpha_1 = \alpha_2 \equiv \alpha$, $\beta \equiv \frac{3}{4}\alpha$, $\psi_2 = -\psi_1 = \psi > 0$ and $\kappa = \sin^2\psi = O(\alpha)$ (we avoid the approximation $\kappa = \psi^2$ in order to facilitate comparison with the results for $\kappa \gg \alpha$). Guided by Whitham's (W§17.2) treatment of the unidirectional interaction (for which $\psi_2 = \psi_1 \equiv 0$ and $\alpha_2 \neq \alpha_1$), we invoke the transformation

$$f = \mathscr{D}\log E(\theta_1,\theta_2), \quad \eta = \mathscr{D}^2\log E(\theta_1,\theta_2)+O(\alpha), \quad (5.1a, b)$$

where

$$\theta_n = x\cos\psi+(-)^n z\sin\psi - ct \equiv \xi_n - \tfrac{1}{2}\tau \quad (n = 1, 2), \quad (5.2a)$$

$$k_n = 1+O(\alpha), \quad c_n = 1+\tfrac{1}{2}\alpha+O(\alpha^2) \quad (5.2b, c)$$

and

$$\mathscr{D} = \partial_1+\partial_2, \quad \partial_n = \partial/\partial\theta_n. \quad (5.3a, b)$$

Substituting (5.1a) into (3.5) with $\kappa = O(\alpha)$ yields

$$\alpha[\{E\mathscr{D}-(\mathscr{D}E)\}\mathscr{D}(\mathscr{D}^2-4)E+3\{(\mathscr{D}^2E)^2-(\mathscr{D}E)(\mathscr{D}^3E)\}] - 16\kappa\{E\partial_1\partial_2E-(\partial_1E)(\partial_2E)\}+O(\alpha^2) = 0. \quad (5.4)$$

Following Whitham's procedure, we obtain the solution [cf. W(17.18)]

$$E = 1+E_1+E_2+e^{2\delta}E_1E_2, \quad E_n = \exp(-2\theta_n), \quad (5.5a, b)$$

where

$$\delta = \tfrac{1}{2}\log\{\kappa/(\kappa-3\alpha)\}. \quad (5.6)$$

Substituting (5.5) into (5.1) yields

$$\tfrac{1}{4}\eta = \frac{E_1+E_2+e^{2\delta}(4+E_1+E_2)E_1E_2}{(1+E_1+E_2+e^{2\delta}E_1E_2)^2}. \quad (5.7)$$

Error factors of $1+O(\alpha)$ are implicit in (5.5) and (5.7) and in the subsequent approximations to E and η.

Remarking that $\theta_2-\theta_1 = 2z\sin\psi > 0$, we let $\theta_2\uparrow\infty$ with θ_1 fixed in (5.7) to obtain

$$\eta \sim \operatorname{sech}^2\theta_1 \equiv \eta_1 \quad [\theta_1 = O(1),\ z\uparrow\infty], \quad (5.8)$$

which describes an incoming solitary wave in $z > 0$ (see figure 1). Similarly, we let $\theta_1\downarrow -\infty$ with θ_2 fixed to obtain

$$\eta \sim \operatorname{sech}^2(\theta_2-\delta) \equiv \eta_{2\delta} \quad [\theta_2 = O(1),\ z\uparrow\infty], \quad (5.9)$$

which describes a reflected solitary wave with the negative phase shift δ. [The converse limits, $\theta_1\uparrow\infty$ with θ_2 fixed and $\theta_2\downarrow -\infty$ with θ_1 fixed, yield the images

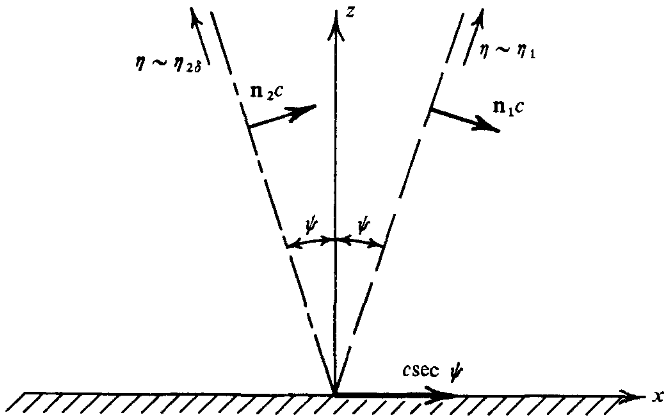

FIGURE 1. The reflexion of the wave $\eta = \eta_1$ (N_1 in § 3) from a rigid wall at $z = 0$. The wave surfaces of constant θ_1 (— — —) and θ_2 (— – – —) advance with the phase velocities $\mathbf{n}_1 c$ and $\mathbf{n}_2 c$, and the interaction is stationary in a reference frame moving parallel to the wall with the speed $c \sec \psi$.

$\eta \sim \eta_2$ and $\eta \sim \eta_{1\delta}$ in $z < 0$, and the solution in the full plane then is equivalent to that of the following section with $\alpha_1 = \alpha_2 = \alpha$ therein. The limits $t \to \pm\infty$ with θ_1 fixed imply $\theta_2 \sim 2ct \sin\psi \sin\phi \sec(\phi - \psi_1)$, where ϕ is the polar angle measured from the $+x$ axis, and are non-uniform in the neighbourhood of the Stokes line $\phi = \frac{1}{2}\pi + \psi_1$ and similarly for $1 \leftrightarrow 2$.]

The interaction is stationary, and therefore optimally resolved, in a reference frame moving with the velocity $c\{\sec\psi, 0\}$. Introducing the co-ordinates

$$\mathscr{x} = x\cos\psi - ct - \tfrac{1}{2}\delta, \quad \mathscr{z} = z \sin\psi, \tag{5.10a, b}$$

such that $\theta_{1,2} = \mathscr{x} + \frac{1}{2}\delta \mp \mathscr{z}$ in (5.7), we obtain (see figure 2)

$$\eta = 4(1 + e^{-\delta}\cosh 2\mathscr{x} \cosh 2\mathscr{z})/(\cosh 2\mathscr{x} + e^{-\delta}\cosh 2\mathscr{z})^2, \tag{5.11}$$

and
$$\eta_{\max} = 4(1+e^{-\delta})^{-1} = 2(1 + \tanh\tfrac{1}{2}\delta) \quad (\mathscr{x} = \mathscr{z} = 0). \tag{5.12}$$

Letting $\alpha/\kappa \downarrow 0$ in (5.6) and (5.7) yields

$$\delta = \tfrac{3}{2}(\alpha/\kappa) + O(\alpha/\kappa)^2 \tag{5.13}$$

and
$$\eta = \operatorname{sech}^2\theta_1 + \operatorname{sech}^2\theta_2 + \delta\{\operatorname{sech}^2\theta_1 \operatorname{sech}^2\theta_2 + \operatorname{sech}^2\theta_1 \tanh\theta_1(1-\tanh\theta_2) + \operatorname{sech}^2\theta_2 \tanh\theta_2(1-\tanh\theta_1)\} + O(\delta^2), \tag{5.14}$$

which is identical to the approximation obtained by setting $\alpha_1 = \alpha_2 = \alpha$ and retaining only the dominant terms in α/κ as $\kappa \downarrow 0$ in (4.4). In brief, (4.4) and (5.7) are matched approximations for the reflexion problem in the overlapping domain $\alpha \ll \kappa \ll 1$.

The preceding solution is formally valid for $\kappa < 3\alpha$, but then (the results are independent of which branch of the logarithm is selected)

$$\delta = \Delta + \tfrac{1}{2}i\pi, \quad \Delta = \tfrac{1}{2}\log\{\kappa/(3\alpha - \kappa)\}, \tag{5.15}$$

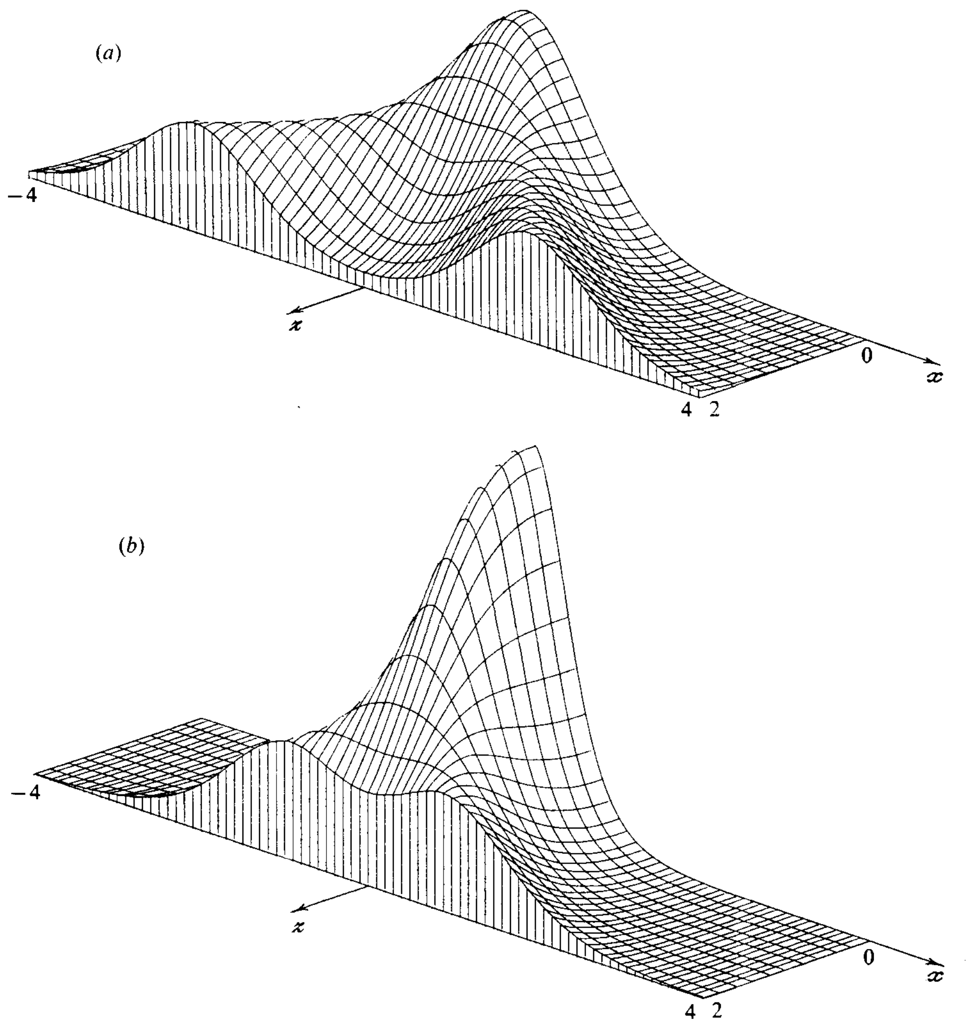

FIGURE 2. The interaction-zone profiles given by (5.10) and (5.11): (*a*) $\delta = 0$; (*b*) $\delta = 2$.

(5.8) remains unchanged, (5.9) describes the singular wave

$$\eta \sim -\operatorname{csch}^2(\theta_2 - \Delta) \quad [\theta_2 = O(1), \quad z \uparrow \infty], \tag{5.16}$$

and (5.11) transforms to

$$\eta = 4(e^{-\Delta}\sinh 2x \cosh 2z - 1)/(\sinh 2x + e^{-\Delta}\cosh 2z)^2, \tag{5.17}$$

where x is obtained by replacing δ by Δ in (5.10*a*). It follows from (5.17) that (see figure 3)

$$\eta < 0 \quad \text{for} \quad x < \tfrac{1}{2}\sinh^{-1}(e^{\Delta}\operatorname{sech} 2z) \equiv x_0(z) \tag{5.18}$$

and
$$\eta = -\infty \quad \text{on} \quad x = -\tfrac{1}{2}\sinh^{-1}(e^{-\Delta}\cosh 2z) \equiv x_\infty(z). \tag{5.19}$$

The hypotheses of weak dispersion and weak nonlinearity obviously fail in the neighbourhood of $x = x_\infty(z)$. The most plausible interpretation of this

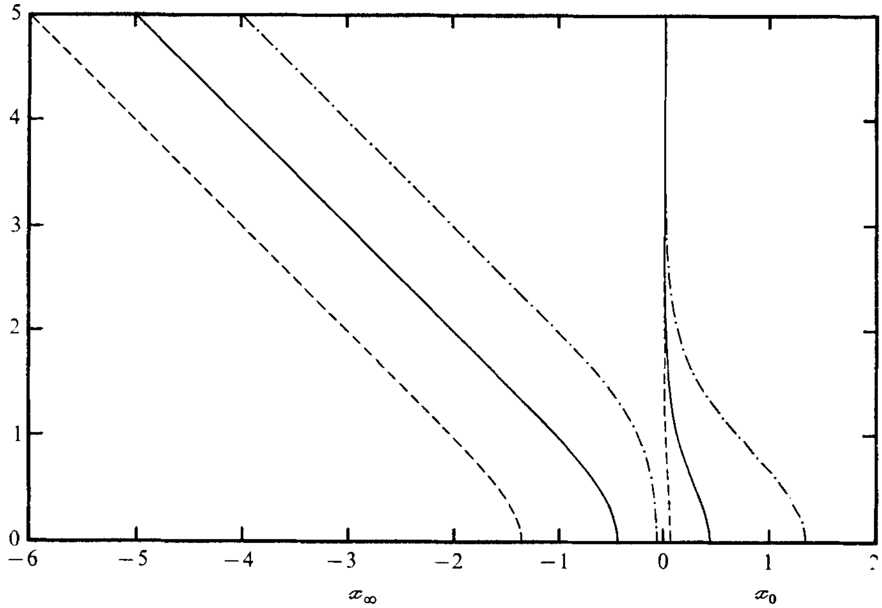

FIGURE 3. The loci of $\eta = 0$ and $\eta = \infty$, $x = x_0(z)$ and $x = x_\infty(z)$, as determined from (5.17)–(5.19). – – –, $\Delta = -2$; ———, $\Delta = 0$; — · —, $\Delta = 2$.

singularity in the present context is that regular reflexion of a solitary wave is impossible for $0 < \kappa < 3\alpha$.

6. Strong asymmetric interaction

We now extend the formulation of § 5 to two solitary waves of unequal strength. Replacing (5.2) and (5.3) by

$$\theta_n = k_n(x \cos \psi_n + z \sin \psi_n - c_n t) + \theta_{0n}, \tag{6.1a}$$

$$\beta^{\frac{1}{2}} k_n = \mathscr{k}_n = (\tfrac{3}{4}\alpha_n)^{\frac{1}{2}}\{1 + O(\alpha)\}, \quad c_n = 1 + \tfrac{1}{2}\alpha_n + O(\alpha^2) \tag{6.1b, c}$$

and

$$\mathscr{D} = k_1 \partial_1 + k_2 \partial_2, \quad \partial_n = \partial/\partial\theta_n \tag{6.2a, b}$$

yields $\alpha[\{E\mathscr{D} - (\mathscr{D}E)\}(\mathscr{D}^3 - 4\mathscr{k}_1^3\partial_1 - 4\mathscr{k}_2^3\partial_2)E + 3\{(\mathscr{D}^2E)^2 - (\mathscr{D}E)(\mathscr{D}^3E)\}]$

$$- 16k_1k_2\kappa\{E\partial_1\partial_2E - (\partial_1E)(\partial_2E)\} + O(\alpha^2) = 0 \tag{6.3}$$

in place of (5.4). The required solution remains of the form (5.5), but with (note that $\mathscr{k}_1^2 = \mathscr{k}_2^2 = \beta = \frac{3}{4}\alpha$ in § 5)

$$\delta = \tfrac{1}{2}\log[\{\kappa - (\mathscr{k}_1 - \mathscr{k}_2)^2\}/\{\kappa - (\mathscr{k}_1 + \mathscr{k}_2)^2\}] \tag{6.4}$$

in place of (5.6). Substituting (5.5) and (6.2) into (5.1) yields

$$\tfrac{1}{4}\eta = \frac{k_1^2E_1 + k_2^2E_2 + (k_1 - k_2)^2E_1E_2 + e^{2\delta}\{(k_1 + k_2)^2 + k_2^2E_1 + k_1^2E_2\}E_1E_2}{(1 + E_1 + E_2 + e^{2\delta}E_1E_2)^2} \tag{6.5}$$

in place of (5.7). An error factor of $1 + O(\alpha)$ is implicit in (6.5) and in the subsequent approximations to η.

Setting $\kappa = 0$ in (6.4) and (6.5) yields the equivalent of W(17.21); setting $k_2 = k_1 = 1$ yields (5.6) and (5.7); letting $\alpha/\kappa \downarrow 0$ with $\alpha \ll \kappa \ll 1$ matches (4.4). Letting $\theta_2 \to \pm\infty$ with θ_1 fixed and similarly for $1 \leftrightarrow 2$ yields [cf. (5.8) and (5.9)]

$$\eta \sim \begin{matrix}\eta_1 \\ \eta_{1\delta}\end{matrix} \quad (\theta_2 \to \pm\infty), \quad \eta \sim \begin{matrix}\eta_2 \\ \eta_{2\delta}\end{matrix} \quad (\theta_1 \to \pm\infty), \tag{6.6a, b}$$

where

$$\eta_n = k_n^2 \operatorname{sech}^2 \theta_n, \quad \eta_{n\delta} = k_n^2 \operatorname{sech}^2(\theta_n - \delta). \tag{6.7a, b}$$

The interaction is stationary, and the resolution of incoming and outgoing waves among η_1, $\eta_{1\delta}$, η_2 and $\eta_{2\delta}$ is optimal, in a reference frame, hereinafter R_*, moving with the velocity $\mathbf{c}_* = c_*\{\cos\psi_*, \sin\psi_*\}$, which is determined by the requirements that its component normal to a surface of constant θ_n be equal to c_n for both $n = 1$ and $n = 2$ or, equivalently (cf. Snell's law),

$$\frac{\cos(\psi_1 - \psi_*)}{c_1} = \frac{\cos(\psi_2 - \psi_*)}{c_2} = \frac{1}{c_*}. \tag{6.8}$$

Solving (6.8) for c_* and ψ_* yields (after some trigonometric reduction)

$$c_* = \tfrac{1}{2}(c_1 + c_2)(\sec^2\psi + \epsilon^2 \csc^2\psi)^{\frac{1}{2}} \tag{6.9}$$

and

$$\psi_* = \tfrac{1}{2}(\psi_1 + \psi_2) + \tan^{-1}(\epsilon \cot\psi), \tag{6.10}$$

where

$$\psi = \tfrac{1}{2}(\psi_2 - \psi_1), \quad \epsilon = (c_2 - c_1)/(c_2 + c_1) \tag{6.11a, b}$$

are relative measures of direction and speed. We assume $\psi > 0$, thereby excluding unidirectional interactions (for which R_* does not exist); the results for this special case are given in W§17.2 and need not be discussed here. We also note that (6.8)–(6.11) are valid for arbitrary (real) values of c_n and ψ_n, even though $\psi = O(\alpha^{\frac{1}{2}})$ and $\epsilon = \frac{1}{4}(\alpha_2 - \alpha_1) + O(\alpha^2)$ in the present context.

We now align the x axis with $\mathbf{c}_*$, such that $\psi_* \equiv 0$,

$$\tan\psi_1 = -(\tan\psi + \epsilon\cot\psi)/(1-\epsilon) \tag{6.12a}$$

and

$$\tan\psi_2 = (\tan\psi - \epsilon\cot\psi)/(1+\epsilon) \tag{6.12b}$$

(note that $\psi_{1,2}$ are small if and only if $|\epsilon| \ll \psi$), and transform (6.1a) to

$$\Theta_n \equiv (c_*/k_n c_n)(\theta_n - \theta_{0n}) = x - c_* t + z\tan\psi_n. \tag{6.13}$$

The geometrical implications of the asymptotic limits of (6.6) in R_* then may be inferred from the signs of $\tan\psi_n$ and

$$\Theta_2 - \Theta_1 = z(\tan\psi_2 - \tan\psi_1) = (c_*^2/c_1 c_2)\, z \sin 2\psi \tag{6.14}$$

(cf. $\theta_2 - \theta_1 = 2z\sin\psi$ in §5, wherein $c_1 = c_2 = c_*\cos\psi$). The limits $\theta_2 \to \pm\infty$ with θ_1 fixed, as in (6.6a), imply $z \to \pm\infty$ (signs vertically ordered), and the corresponding wave surfaces will appear downstream/upstream of (outgoing/incoming to) an observer in R_* if $\tan\psi_1 > 0$ or upstream/downstream if $\tan\psi_1 < 0$. Similarly, $\theta_1 \to \pm\infty$ with θ_2 fixed, as in (6.6b), implies $z \to \mp\infty$, and the corresponding wave surfaces will appear upstream/downstream if $\tan\psi_2 > 0$ or downstream/upstream if $\tan\psi_2 < 0$. The sign of $\tan\psi_n$ is determined by (6.12) and depends on the ratio $\epsilon\cot^2\psi \doteqdot \epsilon/\psi^2$. The scale of the interaction zone if ϵ

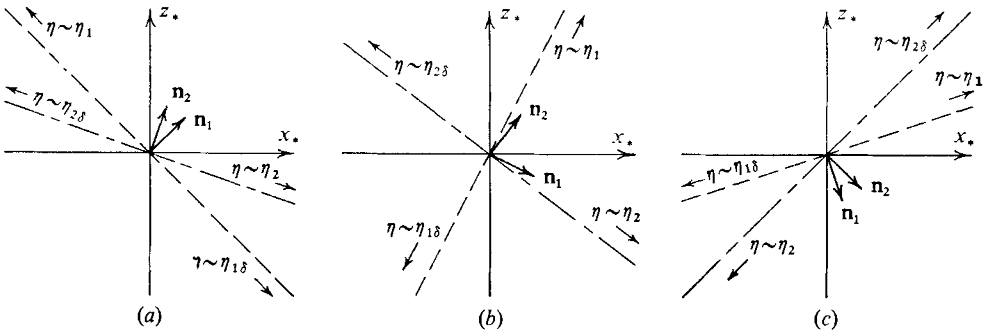

FIGURE 4. The asymptotic (scattering) limits described by (6.6) and (6.15) in the co-ordinates $x_* = x - c_* t$ and $z_* = z \tan \psi$ (the z co-ordinate is stretched for graphical resolution): (a) $\epsilon < -\tan^2\psi$; (b) $|\epsilon| < \tan^2\psi$; (c) $\epsilon > \tan^2\psi$. The interaction is stationary in this reference frame, and incoming/outgoing waves appear downstream/upstream of $x_*, z_* = O(1)$. The broken lines are surfaces of constant θ_1 (— — —) and θ_2 (— – —).

$\theta_1 \sim$	$\theta_2 \sim$	$z \sim$	$\eta \sim$	$x - c_* t$		
				$\epsilon < -\kappa$	$\lvert\epsilon\rvert < \kappa$	$\epsilon > \kappa$
$O(1)$	$\pm\infty$	$\pm\infty$	η_1 $\eta_{1\delta}$	$\mp\infty$	$\pm\infty$	$\pm\infty$
$\pm\infty$	$O(1)$	$\mp\infty$	η_2 $\eta_{2\delta}$	$\pm\infty$	$\pm\infty$	$\mp\infty$

TABLE 1. The asymptotic limits associated with (6.15) and figure 4

and ψ^2 are of the same magnitude is $l = d/\beta^{\frac{1}{2}}$ in the x direction and $l/\alpha^{\frac{1}{2}} \sim d/\alpha$ in the z direction; the scale for $\psi^2 \ll \epsilon$ is essentially that for a unidirectional interaction (W§17.2).

The preceding limits are summarized in table 1, and representative results are sketched in figure 4. Summing up, we find that (6.5) describes the scattering (incoming $\rightarrow$ outgoing) interactions

$$\{\eta_{1\delta}, \eta_2\} \rightarrow \{\eta_1, \eta_{2\delta}\} \quad (\epsilon < -\tan^2\psi), \tag{6.15a}$$

$$\{\eta_1, \eta_2\} \rightarrow \{\eta_{1\delta}, \eta_{2\delta}\} \quad (|\epsilon| < \tan^2\psi), \tag{6.15b}$$

$$\{\eta_1, \eta_{2\delta}\} \rightarrow \{\eta_{1\delta}, \eta_2\} \quad (\epsilon > \tan^2\psi). \tag{6.15c}$$

We remark that the interactions described by (6.15*a*, *c*), but not (6.15*b*), are *phase-conserving* (the sum of the phases of the incoming waves is equal to the sum of the phases of the outgoing waves).

The marginal cases $\epsilon = \mp \tan^2\psi$ invite special comment. Setting $\epsilon = -\tan^2\psi$ in (6.9) and (6.10) yields $c_* = c_1$ and $\psi_* = \psi_1 \equiv 0$: an observer in R_* then perceives $\eta \sim \eta_1/\eta_{1\delta}$ on his left/right and $\eta \sim \eta_2/\eta_{2\delta}$ upstream/downstream. Similarly, $\epsilon = \tan^2\psi$ implies $\psi_* = \psi_2 \equiv 0$ and $c_* = c_2$, and an observer in R_* perceives $\eta \sim \eta_{2\delta}/\eta_2$ on his left/right and $\eta \sim \eta_1/\eta_{1\delta}$ upstream/downstream.

The solution (6.5) is regular for all $\mathbf{x}$ and t if and only if δ is real, i.e. if either

$$\text{(i)} \quad 0 \leqslant \kappa < (k_2 - k_1)^2 \equiv \kappa_- \quad (\delta < 0) \tag{6.16a}$$

or $$\text{(ii)}\ 1 > \kappa > (k_2 + k_1)^2 \equiv \kappa_+ \quad (\delta > 0); \tag{6.16b}$$

it is singular [cf. (5.14)] if

$$\text{(iii)}\ \kappa_- < \kappa < \kappa_+ \quad (\arg \delta = \tfrac{1}{2}\pi). \tag{6.16c}$$

We remark that: (6.15*b*) holds throughout the regular regime (ii), so that all interactions in this regime, which includes all regular reflexions, are non-phase-conserving; (6.15*a*) holds throughout the regular regime (i) if $1 < \alpha_1/\alpha_2 < 4$, but the transition from (6.15*a*) to (6.15*b*) occurs within that regime if $\alpha_1/\alpha_2 > 4$; (6.15*c*) holds throughout (i) if $1 < \alpha_2/\alpha_1 < 4$, but the transition from (6.15*b*) to (6.15*c*) occurs within that regime if $\alpha_2/\alpha_1 > 4$.

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